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On Equivalence of Parameterized Surfaces with Respect to Linear Change of Parameters

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ABSTRACT

The equivalence problem of parameterized surfaces with respect to linear changes of parameters is considered. Separating systems of invariants and uniqueness theorem are offered. The field of invariant differential rational functions over the constant field is described as a differential field by giving a finite system of generators.

Keywords: Parameterized Surface, Linear Change of Parameters, Differential Rational Invariant.

1. Introduction

To set up our investigation problems let us consider the following typical geometric problem.

Let n, m be natural numbers, H be a subgroup of the affine group $GL(n,\mathbb{R}) \propto \mathbb{R}^n$, $\widehat{G}$ be a subgroup of $Diff(\mathbf{B})$ the group of diffeomorphisms of an open unit ball $\mathbf{B} \subset \mathbb{R}^m$ and $\mathbf{u} : \mathbf{B} \to \mathbb{R}^n$ be a parameterized surface, where $\mathbb{R}$ is the field of real numbers, $\mathbf{u}$ is considered to be infinitely smooth, written in row form.

Definition 1.1. *A function $f^{\partial}(\mathbf{u}(\mathbf{t}))$ of $\mathbf{u}(\mathbf{t}) = (u_1(\mathbf{t}), ..., u_n(\mathbf{t}))$ and its finite number of derivatives relative to $\frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_m}$ is said to be invariant (more exactly, $(\widehat{G}, H)$- invariant) if the equality*

$$f^{\partial}(\mathbf{u}(\mathbf{t})) = f^{\delta}(\mathbf{u}(\mathbf{s}(\mathbf{t}))h + h_0)$$

holds for any $\mathbf{s} \in \widehat{G}$, $(h, h_0) \in H$ and $\mathbf{t} \in \mathbf{B}$, where $\mathbf{s}(\mathbf{t}) = (s_1(\mathbf{t}), ..., s_m(\mathbf{t}))$, $\delta^i = \frac{\partial}{\partial s_i}$.

It should be noted that

$$\frac{\partial}{\partial t_i} = \sum_{j=1}^{m} \frac{\partial s_j(\mathbf{t})}{\partial t_i} \frac{\partial}{\partial s_i},$$

i.e. $\delta = g^{-1}\partial$, where g is matrix with the elements $g^i_j = \frac{\partial s_j(\mathbf{t})}{\partial t_i}$, $i, j = 1, ..., m$, and ∂ (δ) is the column vector with the "coordinates" $\frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_m}$ (respectively, $\frac{\partial}{\partial s_i}, ..., \frac{\partial}{\partial s_m}$). Note also that if $s_j(\mathbf{t})$ are linear over $\mathbb{R}$ with respect to $\mathbf{t}$ then $g \in GL(m, \mathbb{R})$.

Such invariants are important in the equivalence problem of parameterized surfaces relative to corresponding motion groups of vector spaces and gauge transformations. Therefore it was the main objective of study for numerous papers with different geometrical methods appropriate to the motion group H.

To represent the above considered problem in a different way let $\mathbf{t}$ run $\mathbf{B}$ and $F = C^{\infty}(\mathbf{B})$ be the differential ring of infinitely smooth functions relative to differential operators $\frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_m}$. Every infinitely smooth parameterized surface $\mathbf{u} : \mathbf{B} \to \mathbb{R}^n$ can be regarded as an element of differential module $(F^n; \partial^1, \partial^2, ..., \partial^m)$, with the coordinate-wise action of $\partial^i = \frac{\partial}{\partial t_i}$ on elements of F^n. If elements of this module are written in row form the above transformations look like

$$\mathbf{u} = (u_1, ..., u_n) \mapsto \mathbf{u}h + h_0, \quad \partial \mapsto g^{-1}\partial,$$

where g is a matrix with elements $g^i_j = \frac{\partial s_j(\mathbf{t})}{\partial t_i}$ $i, j = 1, ..., m$, $(h, h_0) \in H$, $\mathbf{s} \in \widehat{G}$. Therefore the following algebraic analogue of the above problem is natural.

Let $(F; \partial)$ be any differential field, where ∂ is a column-vector of commuting system of differential operators $\partial^1, ..., \partial^m$ of F,

$$C = \{a \in F : \partial^i a = 0 \text{ for } i = 1, ..., m\}$$

be its constant field, H be a subgroup of $GL(n, C) \propto C^n$ and

$$GL^{\partial}(m, F) = \{g \in GL(m, F) : \partial^i g^j_k = \partial^j g^i_k \text{ for } i, j, k = 1, ..., m\}.$$

One can easily verify that (F, δ) also is a differential ring with the commuting system of differential operators $\delta^1, \delta^2, ..., \delta^m$ whenever $\delta = g^{-1}\partial$, $g \in GL^{\partial}(m, F)$. This transformation is an analogue of gauge transformation (change of variables) for abstract differential rings. In general the set $GL^{\partial}(m, F)$ is not a group with respect to the ordinary product of matrices as far as it is not closed with respect to that product. But by the use of it a natural groupoid (Weinstein, 1996) can be constructed with the base $\{g^{-1}\partial : g \in GL^{\partial}(m, F)\}$.

Let G stand for a sub-groupoid of $GL^{\partial}(m, F)$, $x_1, ..., x_n$ be differential algebraic independent variables over F and $\mathbf{x}$ stand for the row vector with coordinates $x_1, ..., x_n$. We use the following notations : $C[\mathbf{x}]$ is the ring of polynomials in $x_1, ..., x_n$ (over C), $C(\mathbf{x})$ is the field of rational functions in $\mathbf{x}$, $C\{\mathbf{x}, \partial\}$ is the ring of ∂-differential polynomial functions in $\mathbf{x}$ and $C\langle \mathbf{x}, \partial \rangle$ is the field of ∂-differential rational functions in $\mathbf{x}$ over C.

Definition 1.2. *An element $f^{\partial}\langle \mathbf{x} \rangle \in C\langle \mathbf{x}; \partial \rangle$ is called to be (G, H)- invariant (G- invariant) if the equality*

$$f^{g^{-1}\partial}\langle \mathbf{x}h + h_0 \rangle = f^{\partial}\langle \mathbf{x} \rangle$$

$$(\textit{respectively, } f^{g^{-1}\partial}\langle \mathbf{x} \rangle = f^{\partial}\langle \mathbf{x} \rangle)$$

holds for any $g \in G$, $(h, h_0) \in H$ (respectively, for any $g \in G$).

Let $C\langle \mathbf{x}; \partial \rangle^{(G,H)}$ (respectively, $C\langle \mathbf{x}; \partial \rangle^{G}$) stand for the field of all such (G, H) (respectively, G)- invariant rational functions.

The description of the field $C\langle \mathbf{x}; \partial \rangle^{(G,H)}$ for different (G, H) is of great interest due to its relation with the problem of equivalency of surfaces in different geometries. In differential geometry usually all differential functional (not compulsory differential rational) invariants of x are considered and they are investigated by geometric methods. In this paper we describe $C\langle \mathbf{x}; \partial \rangle^{G}$ for some subgroups G of $GL(m, C)$ by pure algebraic means. To the best knowledge of the author even in classical cases analogies of these results have not been obtained for arbitrary n. For the used notions of differential field in introduction one can see (Kolchin, 1973), (Bekbaev, 2006).

2. Preliminaries

In this section we deal with some notions and results which will be used in future. These notions and results can be found in detail in (Bekbaev, 2012),(Bekbaev, 2010). Let F stand for a field of characteristic zero.

For a positive integer n let I_n stand for all row n-tuples with nonnegative integer entries with the following linear order: $\beta = (\beta_1, \beta_2, ..., \beta_n) < \alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ if and only if $|\beta| < |\alpha|$ or $|\beta| = |\alpha|$ and $\beta_1 > \alpha_1$ or $|\beta| = |\alpha|$, $\beta_1 = \alpha_1$ and $\beta_2 > \alpha_2$, et cetera , where $|\alpha|$ stands for $\alpha_1 + \alpha_2 + ... + \alpha_n$. We consider in I_n component-wise addition and subtraction (when the result is in I_n), for example, $\alpha + \beta = (\alpha_1 + \beta_1, ..., \alpha_n + \beta_n)$. It is clear that for $\alpha, \beta, \gamma \in I_n$ one has $\alpha < \beta$ if and only if $\alpha + \gamma < \beta + \gamma$. We write $\beta \ll \alpha$ if $\beta_i \leq \alpha_i$ for all $i = 1, 2, ..., n$, $\left(\begin{array}{c} \alpha \\ \beta \end{array} \right)$ stands for $\frac{\alpha!}{\beta!(\alpha-\beta)!}$, $\alpha! = \alpha_1!\alpha_2!...\alpha_n!$ Also we consider the following direct (or tensor) product $I_n \times I_m \longrightarrow I_{n+m}$ as follows. If $\alpha \in I_n$, $\beta \in I_m$ then $\alpha \times \beta = (\alpha, \beta) \in I_{n+m}$

Let n and n' be any fixed nonnegative integers (In the case of $n = 0$ it is assumed that $I_n = \{0\}$).

For any nonnegative integer numbers p, p' let $M_{n',n}(p', p; F) = M(p', p; F)$ stand for all "$p' \times p$" size matrices $A = (A_\alpha^{\alpha'})_{|\alpha|=p, |\alpha'|=p'}$ (α' presents row, α presents column and $\alpha \in I_n, \alpha' \in I_{n'}$). The ordinary size of a such matrix is $\left(\begin{array}{c} p' + n' - 1 \\ n' - 1 \end{array} \right) \times \left(\begin{array}{c} p + n - 1 \\ n - 1 \end{array} \right)$. Over such kind matrices we introduce the "symmetric product" as follows:

Definition 2.1. *If $A \in M(p', p; F)$ and $B \in M(q', q; F)$ then $A \bigodot B = C \in M(p' + q', p + q; F)$ such that for any $|\alpha| = p + q$, $|\alpha'| = p' + q'$, where $\alpha \in I_n, \alpha' \in I_{n'}$,*

$$C_\alpha^{\alpha'} = \sum_{\beta, \beta'} \left(\begin{array}{c} \alpha \\ \beta \end{array} \right) A_\beta^{\beta'} B_{\alpha-\beta}^{\alpha'-\beta'},$$

where the sum is taken over all $\beta \in I_n, \beta' \in I_{n'}$, for which $|\beta| = p$, $|\beta'| = p'$, $\beta \ll \alpha$ and $\beta' \ll \alpha'$.

Proposition 2.1. *For the above defined product the following hold true.*

1. *$A \odot B = B \odot A$.*

2. *$(A + B) \odot C = A \odot C + B \odot C$.*

3. *$(A \odot B) \odot C = A \odot (B \odot C)$*

4. *$(\lambda A) \odot B = \lambda(A \odot B)$ for any $\lambda \in F$.*

5. *If $h = (h_1, h_2, ..., h_n) \in M(0, 1; F)$, $v = (v^1, v^2, ..., v^{n'}) \in M(1, 0; F)$,*

then

$$(h^{\odot m})^0_\alpha = m!h^\alpha, \qquad (v^{\odot m})^{\alpha'}_0 = \begin{pmatrix} m \\ \alpha' \end{pmatrix} v^{\alpha'},$$

where h^α *stands for* $h_1^{\alpha_1} h_2^{\alpha_2} ... h_n^{\alpha_n}$, $\begin{pmatrix} m \\ \alpha' \end{pmatrix} = \frac{m!}{\alpha'!}$, *and* $A^{\odot k}$ *means the k-th power of matrix A with respect to the symmetric product.*

In future (F, ∂) will stand for a field F with the given linear independent commuting system $\partial = (\partial^1, ..., \partial^m)$ of differential operators of F. Its constant field will be denoted by C.

Proposition 2.2. *The following are true.*

1. $\partial \odot (A \odot B) = (\partial \odot A) \odot B + A \odot (\partial \odot B)$.

2. $\partial \odot (A(p', q)B(q, p)) = (\partial \odot A)B + \frac{1}{q+1}(A \odot g)(\delta \odot B)$ *whenever*

$$\delta^k (B_\alpha^{\alpha' - e_k}) = \frac{\alpha'_k}{q+1} (\delta \odot B)^{\alpha'}_\alpha$$

for any $k = 1, 2, ..., m$, $\alpha' \in I_m$ *and* $|\alpha| = p$, *where* $\delta = g^{-1}\partial$, $g \in GL^\partial(m, F)$.

3. *For any* $g \in Mat(1, 1; F)$ *the equalities* $gA_1 \odot gA_2 \odot ... \odot gA_k =$ $\frac{g^{\odot k}}{k!}(A_1 \odot A_2 \odot .... \odot A_k), \quad B_1 g \odot B_2 g \odot ... \odot B_k g = (B_1 \odot B_2 \odot .... \odot B_k)\frac{g^{\odot k}}{k!}$ *are held and if g is not singular then the inverse of* $\frac{g^{\odot k}}{k!}$ *is* $\frac{(g^{-1})^{\odot k}}{k!}$.

4. $\partial \odot (A \frac{1}{k!} \delta^{\odot k}) = (\partial \odot A) \frac{1}{k!} \delta^{\odot k} + (A \odot g) \frac{1}{(k+1)!} \delta^{\odot k+1}$, *where* $\delta = g^{-1}\partial$, $g \in GL^\partial(m; F)$.

5. *For* $g \in GL^\partial(m; F)$ *the condition* $g \in GL(m; C)$ *is equivalent to* $\partial \odot g = 0$.

Further let $* : GL(m, F) \to GL(m, F)$ be a group anti-homomorphism, that is $(g_1 g_2)^* = g_2^* g_1^*$ for any $g_1, g_2 \in GL(m, F)$ and E stand for a square matrix order m over F. Consider the following subgroups of $GL(m, C)$

$$G = \{g \in GL(m, C) : \ g^* E g = E\}, \ SG = \{g \in SL(m, C) : \ g^* E g = E\}$$

Note that all classical subgroups of $GL(m, C)$ are in such form. For example, it is easy to see that if $* : g \mapsto g^{-1}$, E is the identity matrix then $G = GL(m, C)$

and $SG = SL(m, C)$, if $* : g \mapsto g^T$, E the identity matrix then $G = O(m, C)$ and $SG = SO(m, C)$, where T stands for the transpose.

Theorem 2.1. *Let $\delta = (\delta^1, \delta^2, ..., \delta^m)$ be any system of commuting differential operators with the same constant field C, $X \in GL^{\partial}(m, F)$, $Y \in GL^{\delta}(m, F)$ be elements such that $X^{-1}\partial = Y^{-1}\delta$ on F. Then*

a. $(X^{-1} \odot X^{-1})(\partial \odot X) = (Y^{-1} \odot Y^{-1})(\delta \odot Y)$ and $X^*EX = Y^*EY$ if and only if $Y = gX$ for some $g \in G$.

b. $(X^{-1} \odot X^{-1})(\partial \odot X) = (Y^{-1} \odot Y^{-1})(\delta \odot Y)$, $X^*EX = Y^*EY$ and $|X| = |Y|$ if and only if $Y = gX$ for some $g \in SG$, where $|X|$ stands for the determinant of X.

Proof. Note that due to $\partial' = X^{-1}\partial = Y^{-1}\delta = \delta'$ one has $YX^{-1} \in GL^{\delta}(m, F)$. Therefore due to **Proposition 2.2** the condition $YX^{-1} \in GL(m, C)$ is equivalent to

$$0 = \delta \odot (YX^{-1}) = (\delta \odot Y)X^{-1} + \frac{1}{2}(Y \odot Y)(\delta' \odot X^{-1}).$$

But one also has the identity

$$0 = \delta \odot (YY^{-1}) = (\delta \odot Y)Y^{-1} + \frac{1}{2}(Y \odot Y)(\delta' \odot Y^{-1}).$$

Hence $YX^{-1} \in GL(m, C)$ is equivalent to

$$(\delta' \odot Y^{-1})Y = (\delta' \odot X^{-1})X = (\partial' \odot X^{-1})X.$$

The left side of this equalities is equal to $-\frac{1}{2}(Y^{-1} \odot Y^{-1})(\delta \odot Y)$ due to

$$0 = \delta' \odot (Y^{-1}Y) = (\delta' \odot Y^{-1})Y + \frac{1}{2}(Y^{-1} \odot Y^{-1})(\delta \odot Y)$$

and the right side is equal to $-\frac{1}{2}(X^{-1} \odot X^{-1})(\partial \odot X)$. Therefore the relation $YX^{-1} = g \in GL(m, C)$ is equivalent to

$$(X^{-1} \odot X^{-1})(\partial \odot X) = (Y^{-1} \odot Y^{-1})(\delta \odot Y).$$

In the first case $YX^{-1} = g \in G$ is equivalent to $(YX^{-1})^*E(YX^{-1}) = E$ that is to $X^*EX = Y^*EY$. In the second case $YX^{-1} = g \in SG$ is equivalent to $(YX^{-1})^*E(YX^{-1}) = E$ and $|YX^{-1}| = 1$. This means that $YX^{-1} = g \in SG$ is equivalent to $X^*EX = Y^*EY$ and $|X| = |Y|$. □

Note that $Y = g^{-1}X$, $X^{-1}\partial = Y^{-1}\delta$ imply $\delta = g^{-1}\partial$ on F. **Theorem 2.1** can be considered as some kind of "Uniqueness up to G (SG)-equivalence" theorem for the solutions in $GL^{\partial}(m, F)$ of the corresponding systems of equations.

Remark 2.1. *The above theorem can be easily generalized for subgroups defined by family of anti-isomorphisms of $GL(m, F)$.*

3. Equivalence of Parameterized Surfaces, Uniqueness, Invariant Differential Rational Functions

Definition 3.1. *We say that pairs $(\partial, \boldsymbol{a}), (\delta, \boldsymbol{b})$, where $\boldsymbol{a}, \boldsymbol{b} \in F^n$, $\delta = (\delta^1, ..., \delta^m)$ is any system of commuting differential operators on F with the same constant field C, are (G, H) equivalent if*

1. $C\langle \boldsymbol{a}; \partial\rangle = C\langle \boldsymbol{b}; \delta\rangle$.

2. There exists an isomorphism J of ∂-differential field $C\langle \boldsymbol{a}; \partial\rangle$ and δ-differential field $C\langle \boldsymbol{b}; \delta\rangle$ such that $J(\boldsymbol{a}) = \boldsymbol{b} = \boldsymbol{a}h + h_0$ for some $(h, h_0) \in H$.

3. There exists $g \in G$ such that $\delta = g^{-1}\partial$ on $C\langle \boldsymbol{a}; \partial\rangle$.

Assume that we have such a nonsingular matrix $M^{\partial}\langle \mathbf{x}\rangle = ((M^{\partial})^i_j\langle x\rangle)_{i,j=\overline{1,m}}$, where $(M^{\partial})^i_j\langle \mathbf{x}\rangle \in C\langle \mathbf{x}, \partial\rangle$ and $\partial^k(M^{\partial})^i_j\langle \mathbf{x}\rangle = \partial^i(M^{\partial})^k_j\langle \mathbf{x}\rangle$ for $i, j, k = \overline{1, m}$, that

$$M^{g^{-1}\partial}\langle \mathbf{x}h + h_0\rangle = g^{-1}M^{\partial}\langle \mathbf{x}\rangle$$

for any $g \in G$ and $(h, h_0) \in H$. In this case $C\langle \mathbf{x}, \partial\rangle^{(G,H)}$ is a differential field with respect to the commuting system of differential operators $\delta^1, ..., \delta^m$, where $\delta = M^{\partial}\langle \mathbf{x}\rangle^{-1}\partial$

Therefore one can try to find nonsingular matrix $M^{\partial}\langle \mathbf{x}\rangle$ and then describe the field $C\langle \mathbf{x}, \partial\rangle^{(G,H)}$ as δ- differential field over C. In the following theorem it is assumed that we have such matrix $M^{\partial}\langle \mathbf{x}\rangle$ for G (SG), $H = \{1\}$ case, $M_0 = \{\mathbf{a} \in F^n : |M^{\partial}\langle \mathbf{a}\rangle| \neq 0\}$.

Theorem 3.1. *If $\delta = (\delta^1, ..., \delta^m)$ is any system of commuting differential operators on F with the same constant field C, $\mathbf{a}$ is any element of M_0 then*

1. Pairs $(\partial, \mathbf{a}), (\delta, \mathbf{a})$ are G equivalent if and only if the following equalities are true.

a. $C\langle \mathbf{a};\delta\rangle = C\langle \mathbf{a};\partial\rangle$

b. $M^{\partial}\langle \mathbf{a}\rangle^{-1}\partial = M^{\delta}\langle \mathbf{a}\rangle^{-1}\delta$ *on* $C\langle \mathbf{a};\partial\rangle$

c. $(M^{\partial}\langle \mathbf{a}\rangle^{-1}\odot M^{\partial}\langle \mathbf{a}\rangle^{-1})(\partial\odot M^{\partial}\langle \mathbf{a}\rangle) = (M^{\delta}\langle \mathbf{a}\rangle^{-1}\odot M^{\delta}\langle \mathbf{a}\rangle^{-1})(\delta\odot M^{\delta}\langle \mathbf{a}\rangle)$

d. $M^{\partial}\langle \mathbf{a}\rangle^{*}EM^{\partial}\langle \mathbf{a}\rangle = M^{\delta}\langle \mathbf{a}\rangle^{*}EM^{\delta}\langle \mathbf{a}\rangle$.

2. Pairs $(\partial,\mathbf{a}),(\delta,\mathbf{a})$ *are SG equivalent if and only if the following equalities are true.*

a. $C\langle \mathbf{a};\delta\rangle = C\langle \mathbf{a};\partial\rangle$

b. $M^{\partial}\langle \mathbf{a}\rangle^{-1}\partial = M^{\delta}\langle \mathbf{a}\rangle^{-1}\delta$ *on* $C\langle \mathbf{a};\partial\rangle$

c. $(M^{\partial}\langle \mathbf{a}\rangle^{-1}\odot M^{\partial}\langle \mathbf{a}\rangle^{-1})(\partial\odot M^{\partial}\langle \mathbf{a}\rangle) = (M^{\delta}\langle \mathbf{a}\rangle^{-1}\odot M^{\delta}\langle \mathbf{a}\rangle^{-1})(\delta\odot M^{\delta}\langle \mathbf{a}\rangle)$

d. $M^{\partial}\langle \mathbf{a}\rangle^{*}EM^{\partial}\langle \mathbf{a}\rangle = M^{\delta}\langle \mathbf{a}\rangle^{*}EM^{\delta}\langle \mathbf{a}\rangle$

e. $|M^{\partial}\langle \mathbf{a}\rangle| = |M^{\delta}\langle \mathbf{a}\rangle|$.

Proof. If $(\partial,\mathbf{a}),(\delta,\mathbf{a})$ are G-equivalent then $\delta = g^{-1}\partial$ for some $g\in G$ on $C\langle a;\delta\rangle = C\langle a;\partial\rangle$ and

$$M^{\delta}\langle \mathbf{a}\rangle = M^{g^{-1}\partial}\langle \mathbf{a}\rangle = g^{-1}M^{\partial}\langle \mathbf{a}\rangle$$

and therefore

$$M^{\partial}\langle \mathbf{a}\rangle^{-1}\partial = (g^{-1}M^{\partial}\langle \mathbf{a}\rangle)^{-1}(g^{-1}\partial) = M^{\delta}\langle \mathbf{a}\rangle^{-1}\delta,$$

$$M^{\partial}\langle \mathbf{a}\rangle^{*}EM^{\partial}\langle \mathbf{a}\rangle = M^{\delta}\langle \mathbf{a}\rangle^{*}EM^{\delta}\langle \mathbf{a}\rangle.$$

Moreover, according to **Proposition 2.2** and the equality

$$\delta\odot(g^{-1}M^{\partial}\langle \mathbf{a}\rangle) = \frac{g^{-1}\odot g^{-1}}{2}(\partial\odot M^{\partial}\langle \mathbf{a}\rangle)$$

one has

$$(M^{\delta}\langle \mathbf{a}\rangle^{-1}\odot M^{\delta}\langle \mathbf{a}\rangle^{-1})(\delta\odot M^{\delta}\langle \mathbf{a}\rangle) = (M^{\partial}\langle \mathbf{a}\rangle^{-1}g\odot M^{\partial}\langle \mathbf{a}\rangle^{-1}g)(\delta\odot(g^{-1}M^{\partial}\langle \mathbf{a}\rangle)) =$$

$$(M^{\partial}\langle \mathbf{a}\rangle^{-1}\odot M^{\partial}\langle \mathbf{a}\rangle^{-1})\frac{g\odot g}{2}\frac{g^{-1}\odot g^{-1}}{2}\partial\odot M^{\partial}\langle \mathbf{a}\rangle = (M^{\partial}\langle \mathbf{a}\rangle^{-1}\odot M^{\partial}\langle \mathbf{a}\rangle^{-1})\partial\odot M^{\partial}\langle \mathbf{a}\rangle.$$

In $g \in SG$ case one also has

$$|M^{\partial}\langle \mathbf{a}\rangle| = |M^{\delta}\langle \mathbf{a}\rangle|.$$

Vice versa, assume that $M^{\partial}\langle \mathbf{a}\rangle^{-1}\partial = M^{\delta}\langle \mathbf{a}\rangle^{-1}\delta$ on $C\langle \mathbf{a};\delta\rangle = C\langle \mathbf{a};\partial\rangle$ and the equalities

$$(M^{\partial}\langle \mathbf{a}\rangle^{-1} \odot M^{\partial}\langle \mathbf{a}\rangle^{-1})(\partial \odot M^{\partial}\langle \mathbf{a}\rangle) = (M^{\delta}\langle \mathbf{a}\rangle^{-1} \odot M^{\delta}\langle \mathbf{a}\rangle^{-1})(\delta \odot M^{\delta}\langle \mathbf{a}\rangle),$$

$$M^{\partial}\langle \mathbf{a}\rangle^{*} E M^{\partial}\langle \mathbf{a}\rangle = M^{\delta}\langle \mathbf{a}\rangle^{*} E M^{\delta}\langle \mathbf{a}\rangle$$

are true. In this case for $X = M^{\partial}\langle \mathbf{a}\rangle$, $Y = M^{\delta}\langle \mathbf{a}\rangle$ **Theorem 2.1** says that $\delta = g^{-1}\partial$ on $C\langle \mathbf{a};\delta\rangle = C\langle \mathbf{a}; g^{-1}\partial\rangle$ for some $g \in G$, which means that $(\partial, \mathbf{a})$, $(\delta, \mathbf{a})$ are G-equivalent. In $|M^{\partial}\langle \mathbf{a}\rangle| = |M^{\delta}\langle \mathbf{a}\rangle|$ case the same theorem guarantees that $g \in SG$. □

In connection with this result and **Theorem 2.1** the following existence problem arises. Let $A \in Mat(2,1;F)$, $B \in Mat(1,1;F)$ be any matrices over F, and $\partial' = (\partial'_1, ..., \partial'_m)$ be a system of commuting differential operators on F. When does the following system

$$\partial \odot X = \frac{1}{2}(X \odot X)A, \quad X^{*}EX = B, \quad \partial' = X^{-1}\partial$$

have a solution in $GL^{\partial}(m, F)$? A detailed investigation of this question is a nice problem. Here we would like to formulate this problem more precisely in $E \in GL(m, C)$, $X^{*} = X^{-1}$ case.

First of all the matrix should be nonsingular and if X is any solution of this system then X^{*} should satisfy the equality

$$(\partial' \odot X^{*})(X^{*})^{-1} = (\partial' \odot B)B^{-1} - \frac{1}{2}(B \odot I)AB^{-1},$$

where $I \in Mat(1,1;F)$ stands for the ordinary identity matrix. Indeed,

$$\partial' \odot B = \partial' \odot (X^{*}EX) = (\partial' \odot (X^{*}E))X + \frac{1}{2}((X^{*}E) \odot X^{-1})(\partial \odot X) =$$

$$(\partial' \odot X^{*})EX + \frac{1}{2}((X^{*}EX) \odot I)\frac{1}{2}(X^{-1} \odot X^{-1})(\partial \odot X) =$$

$$(\partial' \odot X^{*})(X^{*})^{-1}X^{*}EX + \frac{1}{2}(B \odot I)A = (\partial' \odot X^{*})(X^{*})^{-1}B + \frac{1}{2}(B \odot I)A.$$

In our case it implies that there should be the equality

$$\partial' \odot B = \frac{1}{2}(B \odot I)A - AB$$

Therefore in this case we can formulate the existence problem as follows:

Question. If for A, B the equalities

$$\partial' \odot B = \frac{1}{2}(B \odot I)A - AB, \quad |B| = |E|$$

are valid then does exist $X \in GL^{\partial}(m, F)$ such that

$$\partial \odot X = \frac{1}{2}(X \odot X)A, \quad X^{-1}EX = B, \quad \partial' = X^{-1}\partial?$$

It seems that the answer to this question is positive if one requires $X \in GL^{\partial}(m, \widehat{F})$, where $(\widehat{F}, \partial)$ is an extension of (F, ∂).

Theorem 3.1 indicates that if a function $f^{\partial}(\mathbf{x})$ is G invariant then it is a function of $(M^{\partial}\langle\mathbf{x}\rangle^{-1} \odot M^{\partial}\langle\mathbf{x}\rangle^{-1})(\partial \odot M^{\partial}\langle\mathbf{x}\rangle)$ and $M^{\partial}\langle\mathbf{x}\rangle^{*}EM^{\partial}\langle\mathbf{x}\rangle$ in general sense of function. But if $f^{\partial}(\mathbf{x})$ is a ∂-differential rational function can it be expressed as a ∂-differential rational function of components of

$$(M^{\partial}\langle\mathbf{x}\rangle^{-1} \odot M^{\partial}\langle\mathbf{x}\rangle^{-1})(\partial \odot M^{\partial}\langle\mathbf{x}\rangle), \text{ and } M^{\partial}\langle\mathbf{x}\rangle^{*}EM^{\partial}\langle\mathbf{x}\rangle \text{ ?}$$

The following theorem deals with the later in $G = GL(m, C)$ case.

Theorem 3.2. *The equality*

$$C\langle\mathbf{x}; \partial\rangle^{GL(m,C)} = C\langle\mathbf{x}, (M^{\partial}\langle\mathbf{x}\rangle^{-1} \odot M^{\partial}\langle\mathbf{x}\rangle^{-1})(\partial \odot M^{\partial}\langle\mathbf{x}\rangle); \delta\rangle$$

holds true, where δ stands for $M^{\partial}\langle\mathbf{x}\rangle^{-1}\partial$.

Proof. First of all note that all components of $(\delta \odot M^{\partial}\langle\mathbf{x}\rangle^{-1})M^{\partial}\langle\mathbf{x}\rangle$ are in $C\langle\mathbf{x}; \partial\rangle^{GL(m,C)}$. If $f^{g^{-1}\partial}\langle\mathbf{x}\rangle = f^{\partial}\langle\mathbf{x}\rangle = F((\partial^{\odot k}\mathbf{x})_{k=0,1,\ldots})$ then

$$F((\partial^{\odot k}\mathbf{x})_{k=0,1,\ldots}) = F(((g^{-1}\partial)^{\odot k}\mathbf{x})_{k=0,1,\ldots}) = F((\frac{(g^{-1})^{\odot k}}{k!}\partial^{\odot k}\mathbf{x})_{k=0,1,\ldots}).$$

Since the characteristic of F is zero one has

$$F((\partial^{\odot k}\mathbf{x})_{k=0,1,\ldots}) = F((\frac{(S^{-1})^{\odot k}}{k!}\partial^{\odot k}\mathbf{x})_{k=0,1,\ldots})$$

for any matrix $S = (s^i_j)_{i,j=1,2,\ldots,m}$ with the such variables s^i_j over F that $\partial^k s^i_j = \partial^i s^k_j$ for all $i,j,k = 1,2,..,m$. In particular, this equality is true for $S = M^\partial\langle x\rangle$ and note that all components of $(M^\partial\langle \mathbf{x}\rangle^{-1})^{\odot k}\partial^{\odot k}\mathbf{x}$ are in $C\langle x;\partial\rangle^G$.

Let us show that the same components are in $C\langle \mathbf{x},(\delta\odot M^\partial\langle \mathbf{x}\rangle^{-1})M^\partial\langle \mathbf{x}\rangle;\delta\rangle$. Components of $\delta\mathbf{x} = M^\partial\langle \mathbf{x}\rangle^{-1}\partial\mathbf{x}$ are in $C\langle \mathbf{x},(\delta\odot M^\partial\langle \mathbf{x}\rangle^{-1})M^\partial\langle \mathbf{x}\rangle;\delta\rangle$. Now on induction due to the equality

$$\delta\odot((M^\partial\langle \mathbf{x}\rangle^{-1})^{\odot k}\partial^{\odot k}\mathbf{x}) =$$

$$k((\delta\odot M^\partial\langle \mathbf{x}\rangle^{-1})\odot(M^\partial\langle \mathbf{x}\rangle^{-1})^{\odot k-1})\partial^{\odot k}\mathbf{x}+(k+1)^{-1}(M^\partial\langle \mathbf{x}\rangle^{-1})^{\odot k+1}\partial^{\odot k+1}\mathbf{x} =$$

$$k((\delta\odot M^\partial\langle \mathbf{x}\rangle^{-1})M^\partial\langle \mathbf{x}\rangle\odot I^{\odot k-1})(M^\partial\langle \mathbf{x}\rangle^{-1})^{\odot k}\partial^{\odot k}\mathbf{x}+$$

$$(k+1)^{-1}(M^\partial\langle \mathbf{x}\rangle^{-1})^{\odot k+1}\partial^{\odot k+1}\mathbf{x}$$

one can conclude that the components of $(M^\partial\langle \mathbf{x}\rangle^{-1})^{\odot k+1}\partial^{\odot k+1}\mathbf{x}$ also are in $C\langle \mathbf{x},(\delta\odot M^\partial\langle \mathbf{x}\rangle^{-1})M^\partial\langle \mathbf{x}\rangle;\delta\rangle$. Now to complete the proof it is enough to note that due to $M^\partial\langle \mathbf{x}\rangle^{-1}M^\partial\langle \mathbf{x}\rangle = I$ one has

$$(\delta\odot M^\partial\langle \mathbf{x}\rangle^{-1})M^\partial\langle \mathbf{x}\rangle = -\frac{1}{2}(M^\partial\langle \mathbf{x}\rangle^{-1}\odot M^\partial\langle \mathbf{x}\rangle^{-1})(\partial\odot M^\partial\langle \mathbf{x}\rangle).$$

□

Lemma 3.1. *The system of components of $M^\partial\langle \mathbf{x}\rangle$ is algebraic independent over $C\langle \mathbf{x};\partial\rangle^{GL(m,C)}$.*

Proof. Let $P[(s^i_j)_{i,j=1,2,\ldots,m}]$ be any polynomial over $C\langle x;\partial\rangle^{GL(m,C)}$ for which $P[M^\partial\langle \mathbf{x}\rangle] = 0$. In this case it should remain be true if one substitutes $g\partial$ for ∂ into it. As far as the coefficients of this polynomial are not changed by such transformation and $M^{g\partial}\langle \mathbf{x}\rangle = gM^\partial\langle \mathbf{x}\rangle$, one has $P[gM^\partial\langle \mathbf{x}\rangle] = 0$ for any $g \in GL(m,C)$. Due to zero characteristic of C this implies that $P[(s^i_j)_{i,j=1,2,\ldots,m}] = 0$. □

The lemma allows to find a system of generators of $C\langle \mathbf{x};\partial\rangle^G$ in the following way: Find a system of generators of the field $C\langle \mathbf{x};\partial\rangle^{GL(m,C)}(M^\partial\langle \mathbf{x}\rangle)^G$. Its union with the system of components of $\mathbf{x}$, $(M^\partial\langle \mathbf{x}\rangle^{-1}\odot M^\partial\langle \mathbf{x}\rangle^{-1})(\partial\odot M^\partial\langle \mathbf{x}\rangle)$ one can take as a system of generators of the δ- differential field $C\langle \mathbf{x};\partial\rangle^G$.

Note that among the generators one can have such $f^\partial\langle \mathbf{x}\rangle$ for which the equation $f^\partial\langle \mathbf{x}\rangle = 0$ has only trivial solution, for example, like $f^\partial\langle \mathbf{x}\rangle = \sum_{i=1,\ldots,m}(\partial^i x_1)^2$ for real orthogonal group case.

Note also that the solution set of differential equation $f^{\partial}\langle \mathbf{x}\rangle = 0$, where $f^{\partial}\langle \mathbf{x}\rangle \in C\langle \mathbf{x};\partial\rangle^G$, is invariant with respect to the "G-change of parameters". Therefore any system of δ-generators of $C\langle \mathbf{x};\partial\rangle^G$ may have some fundamental meaning in the physical system which's laws are G-invariant a priori. In connection with it one can ask, for example, whether one can have the system of Maxwell equations (may be by including some equations with trivial solutions) as a system of δ-generators of $C\langle \mathbf{x};\partial\rangle^{(G,H)}$ for some (G,H)?

The next result is on the existence of matrix $M^{\partial}\langle \mathbf{x}\rangle$ with the needed property which can be used for any subgroup of $GL(m,C)$.

Theorem 3.3. *For the matrix $M^{\partial}\langle \mathbf{x}\rangle$ consisting of columns*

$$\partial x_1, f_1^{\partial}\langle \mathbf{x}\rangle^{-1}\partial f_1^{\partial}\langle \mathbf{x}\rangle, ..., f_1^{\partial}\langle \mathbf{x}\rangle^{-\frac{m^k-1}{m-1}}\partial f_{m-1}^{\partial}\langle \mathbf{x}\rangle$$

the equality

$$M^{g^{-1}\partial}\langle \mathbf{x}\rangle = g^{-1}M^{\partial}\langle \mathbf{x}\rangle$$

is true for any $g\in GL(m,C)$, where

$$f_1^{\partial}\langle \mathbf{x}\rangle = |\partial\partial^T x_1|, f_{k+1}^{\partial}\langle \mathbf{x}\rangle = |\partial\partial^T f_k^{\partial}\langle \mathbf{x}\rangle|$$

Proof. Note that if $g\in GL(m,C)$ then for $\delta = g^{-1}\partial$ one has

$$\partial\partial^T x_1 = g\delta\delta^T x_1 g^T$$

Taking the determinant of both sides of the equality we have

$$|\partial\partial^T x_1| = |g|^2|\delta\delta^T x_1|. \qquad (1)$$

, where $|M|$ stands for the determinant of the matrix M. Now repeatedly applying (1) to itself setting x_1 as $y = |\partial\partial^T x_1| = |g|^2|\delta\delta^T x_1|$ to get

$$|\partial\partial^T|\partial\partial^T x_1|| = |g|^{2m+2}|\delta\delta^T|\delta\delta^T x_1||$$

and etcetera. So one can construct

$$f_1^{\partial}\langle \mathbf{x}\rangle = |\partial\partial^T x_1|, f_2^{\partial}\langle \mathbf{x}\rangle = |\partial\partial^T|\partial\partial^T x_1||, ..., f_{k+1}^{\partial}\langle \mathbf{x}\rangle = |\partial\partial^T f_k^{\partial}\langle \mathbf{x}\rangle|, ...$$

for which

$$f_k^{\partial}\langle \mathbf{x}\rangle = |g|^{2m^{k-1}+2m^{k-2}+...+2m+2} f_k^{\delta}\langle \mathbf{x}\rangle.$$

Therefore,

$$f_1^{\delta}\langle \mathbf{x}\rangle^{-\frac{m^k-1}{m-1}}\delta f_k^{\delta}\langle \mathbf{x}\rangle = g^{-1}(f_1^{\partial}\langle \mathbf{x}\rangle^{-\frac{m^k-1}{m-1}}\partial f_k^{\partial}\langle \mathbf{x}\rangle).$$

Consider the matrix $M^{\partial}\langle\mathbf{x}\rangle$ consisting of columns

$$\partial x_1, f_1^{\partial}\langle\mathbf{x}\rangle^{-1}\partial f_1^{\partial}\langle\mathbf{x}\rangle, ..., f_1^{\partial}\langle\mathbf{x}\rangle^{-\frac{m^{m}-1}{m-1}}\partial f_{m-1}^{\partial}\langle\mathbf{x}\rangle.$$

Due to the construction of the matrix one has

$$M^{g^{-1}\partial}\langle\mathbf{x}\rangle = g^{-1}M^{\partial}\langle\mathbf{x}\rangle.$$

□

Example 3.1. *Let us consider $m = 1$ (ordinary differential field) case, $\partial = d$. In this case the previous theorem for $M^{\partial}\langle\mathbf{x}\rangle$ provides dx_1, so $\delta = \frac{1}{dx_1}d$ and*

$$(\delta \odot M^{d}\langle\mathbf{x}\rangle^{-1})M^{d}\langle\mathbf{x}\rangle = -\frac{d^2x_1}{dx_1^2}.$$

Therefore the system $\{\mathbf{x}, \frac{d^2x_1}{dx_1^2}\}$, can be taken as a system of generators for the δ-differential field $C\langle\mathbf{x};\partial\rangle^{C^*}$.

Example 3.2. *Let us consider $m = 2$ case. Even in this case the expressions for entries of the matrix $-(\delta \odot M^{d}\langle\mathbf{x}\rangle^{-1})M^{d}\langle\mathbf{x}\rangle = A$ are quite huge. Here are components of*

$$A = \frac{1}{2}(M^{d}\langle\mathbf{x}\rangle^{-1} \odot M^{d}\langle\mathbf{x}\rangle^{-1})(\partial \odot M^{d}\langle\mathbf{x}\rangle \in Mat(2,1;F),$$

where the following notations are used: $\frac{\partial}{\partial s} := \partial^1$, $\frac{\partial}{\partial t} := \partial^2$, $y := x_1$, $z := y_{ss}y_{tt} - y_{st}^2$, $\Delta := y_s z_t - y_t z_s$.

$$A_{(1,0)}^{(2,0)} = \Delta^{-2}(y_{ss}z_t^2 - 2y_{st}z_s z_t + y_{tt}z_s^2),$$

$$A_{(1,0)}^{(1,1)} = \Delta^{-2}z(-y_{ss}z_t y_t + 2y_{st}(y_s z_t + y_t z_s) - y_{tt}z_s y_s),$$

$$A_{(1,0)}^{(0,2)} = \Delta^{-2}z^2(y_{ss}y_t^2 - 2y_{st}y_s y_t + y_{tt}y_s^2),$$

$$A_{(0,1)}^{(2,0)} = z^{-1}\Delta^{-2}(z_{ss}z_t^2 - 2z_{st}z_s z_t + z_{tt}z_s^2),$$

$$A_{(0,1)}^{(1,1)} = z^{-1}\Delta^{-2}(z(-z_{ss}z_t y_t + 2z_{st}(y_s z_t + y_t z_s) - z_{tt}z_s y_s) - z_s z_t(z_t y_s + z_s y_t)),$$

$$A^{(0,2)}_{(0,1)} = \Delta^{-2}(z(z_{ss}y_t^2 - 2z_{st}y_sy_t + z_{tt}y_s^2) - (y_sz_t - z_sy_t)^2).$$

Let $M^{\partial}\langle \mathbf{x}\rangle$ be any element of $GL^{GL(m,C)}(m, C\langle \mathbf{x}; \partial\rangle)$ for which $M^{g^{-1}\partial}\langle \mathbf{x}\rangle = g^{-1}M^{\partial}\langle \mathbf{x}\rangle$ is true for any $g \in GL(m, C)$.

Theorem 3.4. *For any subgroup G of $GL(m, C)$ the equality*

$$\delta - tr.degC\langle \mathbf{x}; \partial\rangle^G/C = n$$

is true, where δ stands for $M^{\partial}\langle \mathbf{x}\rangle^{-1}\partial$.

Proof. First of all we note that $\delta - tr.degC\langle \mathbf{x}; \partial\rangle/C = \partial - tr.degC\langle \mathbf{x}; \partial\rangle/C = n$ as far as $\delta = M^{\partial}\langle \mathbf{x}\rangle^{-1}\partial$. Due to

$$C\langle \mathbf{x}; \partial\rangle = C\langle \mathbf{x}, (M^{\partial}\langle \mathbf{x}\rangle^{-1} \odot M^{\partial}\langle \mathbf{x}\rangle^{-1})(\partial \odot M^{\partial}\langle \mathbf{x}\rangle), M^{\partial}\langle \mathbf{x}\rangle^{-1}; \delta\rangle$$

to prove the theorem for $G = GL(m, C)$ case it is enough to show that entries of $M^{\partial}\langle \mathbf{x}\rangle^{-1}$ are δ- differential algebraic over $C\langle \mathbf{x}; \partial\rangle^{GL(m,C)}$. All components of $(\delta \odot M^{\partial}\langle \mathbf{x}\rangle^{-1})M^{\partial}\langle \mathbf{x}\rangle = 2A$ are in $C\langle \mathbf{x}; \partial\rangle^{GL(m,C)}$. Therefore one has the equality

$$\delta \odot M^{\partial}\langle \mathbf{x}\rangle^{-1} = 2AM^{\partial}\langle \mathbf{x}\rangle^{-1}.$$

One also has $M^{\partial}\langle \mathbf{x}\rangle^{-1} \in GL^{\delta}(m, C\langle \mathbf{x}; \partial\rangle)$ which implies that all components of $M^{\partial}\langle \mathbf{x}\rangle^{-1}$ are δ-differential algebraic over $C\langle \mathbf{x}; \partial\rangle^{GL(m,C)}$. For any subgroup G we have

$$C\langle \mathbf{x}; \partial\rangle^{GL(m,C)} \subset C\langle \mathbf{x}; \partial\rangle^G \subset C\langle \mathbf{x}; \partial\rangle.$$

This implies that $\delta - tr.degC\langle \mathbf{x}; \partial\rangle^G/C = n$. □

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